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Some Properties of Rank-2 Lattice Rules*

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Abstract. A rank-2 lattice rule is a quadrature rule for the (unit) s-dimensional hypercube, of the form

$$Qf = (1/n_1 n_2) \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \bar{f}(j_1 \mathbf{z}_1/n_1 + j_2 \mathbf{z}_2/n_2),$$

which cannot be re-expressed in an analogous form with a single sum. Here $\bar{f}$ is a periodic extension of f, and $\mathbf{z}_1$, $\mathbf{z}_2$ are integer vectors. In this paper we discuss these rules in detail; in particular, we categorize a special subclass, whose leading one- and two-dimensional projections contain the maximum feasible number of abscissas. We show that rules of this subclass can be expressed uniquely in a simple tricycle form.

1. Introduction.

1.1. *Background to Lattice Rules.* Lattice rules are numerical quadrature rules for integration over an s-dimensional hypercube. They are generalizations of the one-dimensional trapezoidal rule which employ abscissas that lie on an s-dimensional lattice. A well-known and important subclass of lattice rules are the number-theoretic rules of Korobov [7]. There is a large literature devoted to number-theoretic rules, some of which appears in the reference list.

Lattice rules were first explicitly introduced by Sloan [10] and Sloan and Kachoyan [11]. In terms of an s-dimensional integration lattice L which contains the integer lattice $\mathbf{Z}^s$, the corresponding lattice rule is defined by

$$Q_L f = \frac{1}{\nu(Q_L)} \sum_{\mathbf{x} \in A(Q_L)} \bar{f}(\mathbf{x}), \tag{1.1}$$

where $A(Q_L)$ is the set of lattice points contained within the half-open unit cube of integration, and $\nu(Q_L)$ is the number of such points. Here $\bar{f}$ is a periodic continuation of f. In Sloan and Kachoyan [11], many properties of lattice rules were derived, based on definition (1.1) and under the assumption that $\bar{f}$ is continuous.

The theory was developed further in Sloan and Lyness [12], exploiting the more convenient definition (1.2) below. It is almost obvious that when t and n_i are positive integers and the components of $\mathbf{z}_i = (z_i^1, z_i^2, \ldots, z_i^s)$ are integers, the form

$$Qf = \frac{1}{n_1 n_2 \cdots n_t} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_t=1}^{n_t} \bar{f}\left(j_1 \frac{\mathbf{z}_1}{n_1} + j_2 \frac{\mathbf{z}_2}{n_2} + \cdots + j_t \frac{\mathbf{z}_t}{n_t}\right) \tag{1.2}$$

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is a lattice rule. Of course, the abscissas in (1.2) do not necessarily lie in the half-open unit cube, but, because of the periodic property of $\bar{f}$, they may be transferred to the half-open unit cube by subtracting appropriate integer vectors. It can also be shown, with little difficulty, that any lattice rule (1.1) may be expressed in the form (1.2). In fact, the same lattice rule may be expressed in this form in many different ways, i.e., using different selections of the parameters t, n_i, $\mathbf{z}_i$ $(i = 1, 2, \ldots, t)$. We refer to (1.2) as a t-cycle form of the rule Q and to t as this form's cycle number. In general this form is *repetitive*, each distinct point (after transfer to the half-open unit cube) being counted $n_1 n_2 \cdots n_t / \nu(Q)$ times. It is termed a *nonrepetitive* form when the number of points $\nu(Q)$ equals $n_1 n_2 \cdots n_t$.

The following results are established in Sloan and Lyness [12] by the use of finite Abelian group theory.

(i) We define the rank m of a given rule Qf as its minimum possible cycle number; this is the smallest value of t for which Qf may be expressed in the form (1.2). We showed that in this case $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m$ are linearly independent and that $1 \leq m \leq s$.

(ii) When expressed *nonrepetitively* in form (1.2) with $t = m$, where m is the rank, the values of $n_1, n_2, \ldots, n_m$ may be chosen to satisfy

$$n_i \text{ divides } n_{i-1}, \qquad i = 2, \ldots, m. \tag{1.3}$$

In this case the integers $n_1, n_2, \ldots, n_m$ are uniquely determined and $n_m > 1$. We term this set of integers the ***invariants*** of Qf. (In some contexts we extend the invariant list to contain s integers by defining $n_{m+1} = n_{m+2} = \cdots = n_s = 1$.)

(iii) A lattice rule Qf expressed nonrepetitively in the form (1.2) has rank $m = t$ if and only if the denominators $n_1, n_2, \ldots, n_t$ have a nontrivial common factor. Moreover, in this case, if the denominators satisfy (1.3), they are indeed the invariants.

Note that, while for a given rule the rank m and invariants n_i are uniquely defined, there remain many different choices for $\mathbf{z}_i$ $(i = 1, \ldots, m)$.

(iv) If the rank m is equal to the dimension s, then the rule Qf with invariants $n_1, n_2, \ldots, n_s$ is an n_s^s copy of a rule having invariants $n_1/n_s, n_2/n_s, \ldots, n_s/n_s$.

(v) If the s-dimensional lattice rule Qf has rank m and invariants $n_1, n_2, \ldots, n_m$, any s'-dimensional projection of Qf with $s' < s$ has rank $m' \leq m$ and invariants $n_1', n_2', \ldots, n_{m'}'$ which satisfy

$$n_i' \text{ divides } n_i, \qquad i = 1, 2, \ldots, m'. \tag{1.4}$$

Within this classification scheme, a Korobov-Conroy rule and the product-trapezoidal rule have ranks 1 and s, respectively. The present work is motivated by the consideration that rules of intermediate rank, which have not been explicitly considered before, might conceivably perform better in some situations than either of these two familiar types.

1.2. *Scope of This Paper.* The present paper is about rank-2 rules, those having $m = 2$. We deal principally with situations in which the conditions on $\mathbf{z}_i$ and n_i $(i = 1, 2)$ ensure that Qf has favorable projections in a sense specified in the definitions in Section 2 below.

Specifically, we treat an s-dimensional rule for which some or all of its two-dimensional projections have the same number of distinct abscissas as the s-dimensional rule itself, and in addition one or all of its one-dimensional projections also have the maximum feasible number of points. Properties of rule projections, and the concept of a rule with "full" projections of various kinds, are defined and discussed in Section 2. For certain rules of this kind it is possible to write down simple representations in which all quantities are uniquely defined. This theory is developed in Sections 3 to 5, the principal results being Theorems 3.3 and 5.4. Such representations may prove useful for computer searches for cost-effective rules.

2. Rule Projections.

It is generally the case in numerical quadrature that one would expect a more accurate approximation to an integral using a rule that employs more abscissas. A more sophisticated expectation is that among rules using the same number of abscissas, the rule which "spreads these out" more is likely to be the more suitable. One way of effecting this is to try to design rules so that their projections in the different lower-dimensional manifolds use as many points as is feasible. In this paper we look at the structure of rank-2 rules with prescribed conditions (defined below) on their lower-dimensional projections. In a separate paper, we shall deal with the error analysis involved.

We recall the definition of an s'-dimensional projection of an s-dimensional rule

$$Q_s f = \sum_{j=1}^{\nu} w_j f(x_j^1, x_j^2, \ldots, x_j^s). \tag{2.1}$$

The *principal* s'-dimensional projection is the s'-dimensional rule

$$Q_{s'} f = \sum_{j=1}^{\nu} w_j f(x_j^1, x_j^2, \ldots, x_j^{s'}). \tag{2.2}$$

This is obtained by omitting the final $s - s'$ components of each abscissa, thus constructing a rule for the cube $C^{s'}$ from one for C^s. Note that (2.2) may be in repetitive form even if (2.1) is not. Note too that (2.2) is simply one of $s!/s'!(s-s')!$ different s'-dimensional projections of $Q_s f$. Thus the s'-dimensional projection of $Q_s f$ into the space determined by the components $x^{j_1}, x^{j_2}, \ldots, x^{j_{s'}}$, where $1 \leq j_1 < j_2 < \cdots < j_{s'} \leq s$, is

$$Q_{s'} f = \sum_{j=1}^{\nu} w_j f(x_j^{j_1}, x_j^{j_2}, \ldots, x_j^{j_{s'}}). \tag{2.3}$$

A mild reformulation of Theorem 5.1 of Sloan and Lyness [12], summarized in (v) above, follows.

THEOREM 2.1. *Let Q_s be an s-dimensional lattice rule having invariants n_1, $n_2, \ldots, n_s$. Then any s'-dimensional projection $Q_{s'}$ of Q_s is a lattice rule having invariants n_1', $n_2', \ldots, n_{s'}'$, where n_i' divides n_i for $i = 1, 2, \ldots, s'$.*

From this, it follows that

$$\nu(Q_{s'}) = n_1' n_2' \cdots n_{s'}' \leq n_1 n_2 \cdots n_{s'}, \tag{2.4}$$

providing an upper bound on the number of points required by any s'-dimensional projection.

Definition. $Q_{s'}$ is a *full* projection of a lattice rule Q_s with invariants n_1, $n_2, \ldots, n_s$ if $Q_{s'}$ has invariants n_1, $n_2, \ldots, n_{s'}$.

There exist many possible definitions, specifying different selections of projections of Q_s which may be full. For our purposes we shall be able to provide all our results in terms of the following definition pair:

Definition. The s-dimensional rule Q_s, having invariants $n_1, n_2, \ldots, n_s$, is said to have *full principal projections in all dimensions* if the s'-dimensional principal projection $Q_{s'}$ has invariants $n_1, n_2, \ldots, n_{s'}$ for $s' = 1, 2, \ldots, s$.

Definition. The s-dimensional rule Q_s, having invariants $n_1, n_2, \ldots, n_s$, is said to have a *complete set of full projections in all dimensions* if every s'-dimensional projection $Q_{s'}$ has invariants $n_1, n_2, \ldots, n_{s'}$ for $s' = 1, 2, \ldots, s$.

Subsequently, we may suppress the phrase "in all dimensions" if no confusion seems likely.

We now specialize these definitions to rules of rank 2. It follows from the first definition that an s-dimensional rank-2 lattice rule, with invariants n_1, n_2, has full principal projections if and only if

(i) the one-dimensional principal projection has invariant n_1, i.e., is the n_1-panel trapezoidal rule;

(ii) the two-dimensional principal projection has invariants n_1, n_2; and

(iii) each s'-dimensional principal projection has invariants n_1, n_2 for $s' = 3, 4, \ldots, s-1$.

Item (iii) is redundant, since by a double application of Theorem 2.1 the invariants of the s'-dimensional principal projections with $2 < s' < s$ are sandwiched between those of the rule itself and those of the two-dimensional principal projection.

The second condition can be streamlined. In view of (i), the two-dimensional principal projection has first invariant n_1; thus, we may replace (ii) by any condition that ensures that the second invariant is n_2. One such possibility is

(ii)$'$ the two-dimensional principal projection is an n_2^2 copy rule.

Finally, in view of Lemma 6.3 of Sloan and Lyness [12], we may replace (ii)$'$ by (ii)$''$, namely,

(ii)$''$ The abscissa set $A(Q_2)$ of the two-dimensional principal projection contains the abscissa set $A(T_2^{n_2})$ of the n_2^2-point trapezoidal rule as a subgroup.

The following theorem summarizes the preceding discussion.

THEOREM 2.2. *An s-dimensional rank-2 lattice rule, having invariants n_1, n_2, has full principal projections if and only if*

(i) *the one-dimensional principal projection has invariant n_1, i.e., is the n_1-panel trapezoidal rule, and*

(ii)$''$ *the abscissa set $A(Q_2)$ of the principal two-dimensional projection contains the abscissa set $A(T_2^{n_2})$ of the n_2^2-point trapezoidal rule as a subgroup.*

A further application of Theorem 2.1 (and in particular, (2.4)) allows the hypotheses in the theorem to be further weakened, and hence more easily tested. Thus in the following statement it is not necessary to know in advance the rank or invariants of the rule, or even to know in advance that the given form of the rule is nonrepetitive.

THEOREM 2.3. *An s-dimensional lattice rule Qf with $\nu(Q) \leq n_1 n_2$, where $n_1 \geq n_2 > 1$ and n_2 divides n_1, is a rank-2 rule with invariants n_1, n_2 and full principal projections if and only if*

(i) *the one-dimensional principal projection has invariant n_1, and*

(ii)″ *the abscissa set $A(Q_2)$ of the principal two-dimensional projection contains $A(T_2^{n_2})$ as a subgroup.*

A companion theorem, relating to complete sets of projections, may be proved in the identical way.

THEOREM 2.4. *An s-dimensional lattice rule Qf with $\nu(Q) \leq n_1 n_2$, where $n_1 \geq n_2 > 1$ and n_2 divides n_1, is a rank-2 rule with invariants n_1, n_2 and a complete set of full projections if and only if*

(i) *every one-dimensional projection has invariant n_1, and*

(ii)″ *the abscissa set $A(Q_2)$ of every two-dimensional projection contains $A(T_2^{n_2})$ as a subgroup.*

In the next two sections we develop conditions on the rule parameters n_1, n_2, $\mathbf{z}_1$, $\mathbf{z}_2$, that allow us to identify rank-2 rules having full projections in the sense of these theorems.

3. Rank-2 Rules Having Full Principal Projections.

Fundamental to the discussion of rank-2 rules is the question of how to recognize whether the rule forms (3.3) and (3.12) below are repetitive or not. We commence this section with two lemmas needed subsequently in dealing with rank-2 rules. We denote the highest common factor of $a_1, a_2, \ldots, a_k$ by $\gcd(a_1, a_2, \ldots, a_k)$, or simply by $(a_1, a_2, \ldots, a_k)$ when no confusion is likely to arise. Note that $(a, 0) = |a|$ and $(a, b, 0) = (a, b)$.

LEMMA 3.1. *A one-dimensional lattice rule in the (bicycle) form*

$$Q_1 f = \frac{1}{n^2 r} \sum_{j_1=1}^{nr} \sum_{j_2=1}^{n} \bar{f}\left(j_1 \frac{z_1}{nr} + j_2 \frac{z_2}{n}\right) \tag{3.1}$$

has $\nu(Q) = nr$ (and hence is the trapezoidal rule $T_1^{nr} f$) if and only if

$$(z_1, r) = 1 \quad \text{and} \quad (z_1, z_2, n) = 1. \tag{3.2}$$

The proof (not given in detail here) rests on three elementary propositions. With $\{a\}$ denoting the fractional part of a, these are as follows:

(a) If the set of points $\{(j_1 z_1 + j_2 z_2 r)/nr\}$ includes the point $1/nr$, (3.1) is the (nr)-panel trapezoidal rule; otherwise it is not.

(b) The pair of conditions (3.2) coincides with the single condition $(z_1, z_2 r, nr) = 1$.

(c) A necessary and sufficient condition that there exist integers j_1, j_2 such that

$$Aj_1 + Bj_2 = 1 \text{ (modulo } C)$$

is simply $(A, B, C) = 1$.

The second lemma is deeper.

LEMMA 3.2. *Let $Q_2 f$ be the two-dimensional lattice rule*

$$Q_2 f = \frac{1}{n^2} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \bar{f}\left(j_1 \frac{(z_1^1, z_1^2)}{n} + j_2 \frac{(z_2^1, z_2^2)}{n} \right). \tag{3.3}$$

This coincides with the product trapezoidal rule

$$T_2^n f = \frac{1}{n^2} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \bar{f}\left(\frac{(k_1, k_2)}{n} \right) \tag{3.4}$$

if and only if

$$(D, n) = 1, \tag{3.5}$$

where

$$D = D_{1,2} = z_1^1 z_2^2 - z_2^1 z_1^2. \tag{3.6}$$

Proof. If rules (3.3) and (3.4) coincide, each assignment of (k_1, k_2) (modulo n) in (3.4) corresponds to an assignment of (j_1, j_2) (modulo n) in (3.3), the relationship being

$$\begin{aligned} j_1 z_1^1 + j_2 z_2^1 &= k_1 \text{ (modulo } n\text{)}, \\ j_1 z_1^2 + j_2 z_2^2 &= k_2 \text{ (modulo } n\text{)}. \end{aligned} \tag{3.7}$$

By a standard manipulation we obtain

$$j_2 (z_1^1 z_2^2 - z_1^2 z_2^1) = k_2 z_1^1 - k_1 z_1^2 \text{ (modulo } n\text{)}, \tag{3.8}$$

which must have a solution j_2 for each assignment of (k_1, k_2) (modulo n). Setting $(k_1, k_2) = (0, 1)$, we obtain

$$j_2 D = z_1^1 \text{ (modulo } n\text{)}, \tag{3.9}$$

and since this has a solution j_2, it follows that z_1^1 contains (D, n) as a factor. Similarly, by setting $(k_1, k_2) = (1, 0)$ we learn that z_1^2 contains (D, n) as a factor. It follows, if $(D, n) > 1$, that z_1^1, z_1^2, and n all have the nontrivial common factor (D, n), in which case (3.3) is manifestly repetitive and so requires less than n^2 abscissas. Since (3.4) is clearly not repetitive, and requires n^2 abscissas, (3.3) and (3.4) cannot then coincide. This establishes the necessity of condition (3.5).

Conversely, if these rules do not coincide, form (3.3) is repetitive. It is easy to show in this case that two or more values of the pair j_1, j_2 give rise to the abscissa zero. Consider the equation

$$j_1 \mathbf{z}_1 + j_2 \mathbf{z}_2 = \mathbf{0} \text{ (modulo } n\text{)}. \tag{3.10}$$

If $(D, n) = 1$ then the application of Cramer's rule to (3.10) gives

$$j_1 = j_2 = 0 \text{ (modulo } n\text{)}, \tag{3.11}$$

contradicting the immediately preceding statement. Thus, $(D, n) > 1$ unless the two rules coincide. This establishes Lemma 3.2. □

In view of the results quoted in item (ii) in Section 1, any s-dimensional lattice rule with rank 2 has invariants nr, n, with $n > 1$, and may be expressed nonrepetitively in the form

$$Qf = \frac{1}{n^2 r} \sum_{j_1=1}^{nr} \sum_{j_2=1}^{n} \bar{f}\left(j_1 \frac{\mathbf{z}_1}{nr} + j_2 \frac{\mathbf{z}_2}{n} \right). \tag{3.12}$$

On the other hand, an expression of the form (3.12) might be repetitive, and hence not correspond to a lattice rule with rank 2 and invariants nr, n.

THEOREM 3.3. *The lattice rule* (3.12) *is a rank*-2 *rule with invariants* nr, n *and full principal projections if and only if*

$$(z_1^1, r) = 1 \tag{3.13}$$

and

$$(D_{1,2}, n) = 1, \tag{3.14}$$

where the determinant $D_{1,2}$ *is defined in* (3.6) *above.*

Proof. We establish this by showing that these conditions correspond precisely to conditions (i) and (ii)″ of Theorem 2.3. The principal one-dimensional projection of Qf is

$$Q_1 f = \frac{1}{n^2 r} \sum_{j_1=1}^{nr} \sum_{j_2=1}^{n} \bar{f}\left(j_1 \frac{z_1^1}{nr} + j_2 \frac{z_2^1}{n}\right). \tag{3.15}$$

The necessary and sufficient condition that this rule contains nr points is given by Lemma 3.1, namely,

$$(z_1^1, r) = 1, \tag{3.16}$$

$$(z_1^1, z_2^1, n) = 1. \tag{3.17}$$

The principal two-dimensional projection of Q is

$$Q_2 f = \frac{1}{n^2 r} \sum_{j_1=1}^{nr} \sum_{j_2=1}^{n} \bar{f}\left(j_1 \frac{(z_1^1, z_1^2)}{nr} + j_2 \frac{(z_2^1, z_2^2)}{n}\right). \tag{3.18}$$

Because of (3.16), an abscissa of the product trapezoidal rule $T_2^n f$ can arise in (3.18) only if j_1 is a multiple of r. The rule obtained by including only values of j_1 that are multiples of r is

$$\tilde{Q}_2 f = \frac{1}{n^2} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \bar{f}\left(k_1 \frac{(z_1^1, z_1^2)}{n} + k_2 \frac{(z_1^1, z_2^2)}{n}\right). \tag{3.19}$$

According to Lemma 3.2, it coincides with the two-dimensional trapezoidal rule if and only if

$$(D_{1,2}, n) = 1. \tag{3.20}$$

Thus (3.16) and (3.17) correspond to (i) of Theorem 2.3, and (3.20) corresponds to (ii)″. However, condition (3.17) is not needed, as it is a consequence of condition (3.20). Since $\nu(Q)$ clearly cannot exceed $n^2 r$, we may apply Theorem 2.3 with $n_1 = nr$ and $n_2 = n$ to establish the theorem. □

4. Rank-2 Rules Having a Complete Set of Full Projections.

In Section 3 we treated rules having full principal projections. The treatment was based on Theorem 2.3. A parallel treatment of rules having a complete set of projections may be based on Theorem 2.4. In this section we give the analog of Theorem 3.3.

THEOREM 4.1. *The lattice rule* (3.12) *is a rank*-2 *rule with invariants* nr, n *and a complete set of full projections if and only if*

$$(z_1^q, r) = 1, \qquad q = 1, 2, \ldots, s, \tag{4.1}$$

and

$$(D_{p,q}, n) = 1, \qquad 1 \le p < q \le s, \tag{4.2}$$

where

$$D_{p,q} = z_1^p z_2^q - z_2^p z_1^q. \tag{4.3}$$

Proof. This follows immediately by applying the result of Theorem 3.3 to all projections, rather than just to principal projections. □

5. Rank-2 Rules in Tricycle Form.

We continue our treatment of rules of rank 2 having full principal projections by showing that they can be expressed in a convenient tricycle form. We commence with the following lemma.

LEMMA 5.1. *Let Qf be a lattice rule in the nonrepetitive tricycle form*

$$Qf = \frac{1}{n^2 r}\sum_{j=1}^{r}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\bar{f}\left(j\frac{\mathbf{x}}{r} + k_1\frac{\mathbf{y}_1}{n} + k_2\frac{\mathbf{y}_2}{n}\right), \tag{5.1}$$

with $n > 1$. If $(n, r) > 1$, then Qf is of rank 3. If $(n, r) = 1$ then Qf is of rank 2 and has invariants nr, n.

Proof. When $(n, r) > 1$, the denominators in (5.1) have a nontrivial common factor, and the result (iii) of Section 1 confirms that Qf is of rank 3. When $(n, r) = 1$, there is no common factor and the same result indicates that Qf is of rank 2 or less. In this case, the cyclic groups of orders r and n generated respectively by $\mathbf{x}/r$ and $\mathbf{y}_1/n$ (using arithmetic modulo 1) may be combined, in the manner discussed for example in Section 3 of Sloan and Lyness [12], into a single cyclic group of order nr, generated, for instance, by $\mathbf{x}/r + \mathbf{y}_1/n$. In other words, Qf can be re-expressed in the (nonrepetitive) bicycle form

$$Qf = \frac{1}{n^2 r}\sum_{l=1}^{nr}\sum_{k=1}^{n}\bar{f}\left(l\frac{\mathbf{z}_1}{nr} + k\frac{\mathbf{y}_2}{n}\right), \tag{5.2}$$

with $\mathbf{z}_1 = n\mathbf{x} + r\mathbf{y}_1$. In view of result (ii) of Section 1, this rule has rank 2 and invariants nr, n. This establishes the lemma. □

THEOREM 5.2. *Let $(n, r) = 1$, with $n > 1$. The lattice rule (5.1) is a rank-2 rule with invariants nr, n and full principal projections if and only if*

$$(x^1, r) = 1 \tag{5.3}$$

and

$$(\bar{D}_{1,2}, n) = 1, \tag{5.4}$$

where

$$\bar{D}_{p,q} = y_1^p y_2^q - y_1^q y_2^p. \tag{5.5}$$

Proof. The principal one- and two-dimensional projections of Qf are

$$Q_1 f = \frac{1}{n^2 r}\sum_{j=1}^{r}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\bar{f}\left(j\frac{x^1}{r} + k_1\frac{y_1^1}{n} + k_2\frac{y_2^1}{n}\right) \tag{5.6}$$

and

$$(5.7)\qquad Q_2 f = \frac{1}{n^2 r}\sum_{j=1}^{r}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\bar{f}\left(j\frac{(x^1,x^2)}{r} + k_1\frac{(y_1^1,y_1^2)}{n} + k_2\frac{(y_2^1,y_2^2)}{n}\right).$$

The rule obtained from $Q_2 f$ by including only those terms for which $j = r$, is

$$(5.8)\qquad \tilde{Q}_2 f = \frac{1}{n^2}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\bar{f}\left(k_1\frac{(y_1^1,y_1^2)}{n} + k_2\frac{(y_2^1,y_2^2)}{n}\right)$$

and coincides, by virtue of Lemma 3.2 and assumption (5.4), with the product trapezoidal rule $T_2^n f$. Thus $Q_2 f$ includes among its abscissas the abscissas of $T_2^n f$. Trivially, then, $Q_1 f$ includes among its abscissas the point $1/n$. Because of (5.3), it also includes the point $1/r$; and because $(n,r) = 1$ it now follows that $Q_1 f = T_1^{nr} f$. The result now follows from Theorem 2.3. □

As a special case, we obtain the following:

THEOREM 5.3. *Let* $(n,r) = 1$, *with* $n > 1$, *and let* Qf *be a lattice rule given in the tricycle form* (5.1), *with*

$$(5.9)\qquad x^1 = 1 \quad \textit{and} \quad \begin{pmatrix} y_1^1 & y_1^2 \\ y_2^1 & y_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then (5.1) *is nonrepetitive, and* Qf *is a rank*-2 *rule with invariants* nr, n *having full principal projections.*

The next theorem establishes the converse of Theorem 5.3: that every rank-2 rule having invariants nr, n, with $(n,r) = 1$, and also having full principal projections, may be expressed in the tricycle form (5.1), with the leading components of $\mathbf{x}$, $\mathbf{y}_1$, and $\mathbf{y}_2$ satisfying (5.9). The significance of this result lies in the fact, expressed in the last part of the theorem, that $\mathbf{x}$, $\mathbf{y}_1$, and $\mathbf{y}_2$ are thereby essentially uniquely determined. It is this uniqueness property that makes the tricycle form potentially attractive for some applications.

THEOREM 5.4. *Let* Qf *be an* s-*dimensional lattice rule with rank* 2 *and invariants* nr, n *where* $(n,r) = 1$. *If* Qf *has full principal projections, it can be expressed in the tricycle form* (5.1), *with the leading components of* $\mathbf{x}$, $\mathbf{y}_1$, *and* $\mathbf{y}_2$ *satisfying* (5.9). *The remaining components of* $\mathbf{x}$, $\mathbf{y}_1$, *and* $\mathbf{y}_2$ *are uniquely determined modulo* r, n *and* n, *respectively.*

Proof. Because r is prime to n, the abscissa set may be expressed (as discussed in Sloan and Lyness [12, Section 3]) as the direct sum of three cyclic subgroups of orders r, n, and n, respectively:

$$(5.10)\qquad A(Q) = C(r) \oplus C(n) \oplus C(n).$$

Taking $\mathbf{X}/r$, $\mathbf{Y}_1/n$, and $\mathbf{Y}_2/n$ as the generators of the respective subgroups, we obtain the nonrepetitive form

$$(5.11)\qquad Qf = \frac{1}{n^2 r}\sum_{j'=1}^{r}\sum_{k_1'=1}^{n}\sum_{k_2'=1}^{n}\bar{f}\left(j'\frac{\mathbf{X}}{r} + k_1'\frac{\mathbf{Y}_1}{n} + k_2'\frac{\mathbf{Y}_2}{n}\right).$$

Because Qf has full principal projections, the two-dimensional principal projection $Q_2 f$ contains the abscissas $(1,0)/n$ and $(0,1)/n$; and since $Q_2 f$ is nonrepetitive, each of these occurs for exactly one combination of k'_1, k'_2 and j'. Moreover, j' must equal r, because r and n are prime. Let $\mathbf{y}_1/n$ and $\mathbf{y}_2/n$ be the s-dimensional abscissas that arise for exactly those values of the trio k'_1, k'_2, and $j' = r$. Then $\mathbf{y}_1$ and $\mathbf{y}_2$ are uniquely determined modulo n. Further, the one-dimensional principal projection $Q_1 f$ contains the abscissa $1/r$; thus there is at least one s-dimensional abscissa of the form $\mathbf{x}/r$ which projects into $1/r$. Because r and n are prime, a vector of this form can arise only if $k'_1 = k'_2 = n$; thus the vector $\mathbf{x}$ is uniquely determined modulo r.

Now we consider

$$\tilde{Q}f = \frac{1}{n^2 r} \sum_{j=1}^{r} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \bar{f}\left(j\frac{\mathbf{x}}{r} + k_1\frac{\mathbf{y}_1}{n} + k_2\frac{\mathbf{y}_2}{n}\right). \tag{5.12}$$

By Lemma 5.1, this is a rank-2 rule with invariants nr, n. It remains only to show that it coincides with Qf. The cyclic subgroup $C(r)$ in (5.10) may be generated by any element of $C(r)$ which is of order r. One such element is $\mathbf{x}/r$. The group $C(n) \oplus C(n)$ in (5.10) may be generated by any pair $\mathbf{c}_1$, $\mathbf{c}_2$, each of order n, provided they give rise to n^2 distinct elements. One such pair is $\mathbf{y}_1/n$ and $\mathbf{y}_2/n$. Thus $\tilde{Q}f = Qf$, establishing the theorem. □

Remark. We note that Theorems 5.3 and 5.4 give the number ν_p of distinct s-dimensional rank-2 lattice rules having invariants nr, n and full principal projections as

$$\nu_p = r^{s-1} n^{2(s-2)}, \tag{5.13}$$

since the components of $\mathbf{x}$, $\mathbf{y}_1$, and $\mathbf{y}_2$ not fixed by (5.9) may be chosen arbitrarily, modulo r, n, and n, respectively.

We conclude with a result for rank-2 rules having a complete set of full projections.

THEOREM 5.5. *If Qf is a rank-2 rule with invariants nr, n with $(n, r) = 1$, and Qf has a complete set of full projections in all dimensions, then it can be expressed in the tricycle form* (5.1), *where*

$$x^1 = 1 \quad \text{and} \quad \begin{pmatrix} y_1^1 & y_1^2 \\ y_2^1 & y_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.14}$$

$$(x^q, r) = 1, \qquad q = 1, 2, \ldots, s, \tag{5.15}$$

and

$$(\bar{D}_{p,q}, n) = 1, \qquad 0 < p < q \leq s, \tag{5.16}$$

where $\bar{D}_{p,q}$ is given in (5.5) *above. The components of $\mathbf{x}$, $\mathbf{y}_1$, and $\mathbf{y}_2$ are uniquely determined modulo r, n, and n, respectively.*

The conditions here are simply those from Theorem 5.2 applied in all dimensions, combined with those from Theorem 5.3. Note that conditions (5.15) and (5.16) are automatic. That is to say, given the rule Qf satisfying the hypotheses of the theorem, one may express it in this tricycle form in many ways. One *may* use a representation which satisfies (5.14). However, *all* representations satisfy (5.15) and (5.16).

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